Online Appendices for "A Vector Monotonicity Assumption for Multiple Instruments"

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D Further Proofs

D.1 Proof of Proposition 1

I prove Proposition 1 by making use of the following alternative characterization of VM (which is more similar in form to the definition of PM)

Lemma D.1. *Let $\mathcal{Z}$ be non-disjoint. Then VM holds iff for each $j \in \{1 \dots J\}$ there is an ordering $\geq_j$ on $\mathcal{Z}_j$ such that the following holds for all i: for all $z_j, z'_j \in \mathcal{Z}_j$ and $\mathbf{z}_{-j} \in \mathcal{Z}_{-j}$ such that both $(z_j, \mathbf{z}_{-j}) \in \mathcal{Z}$ and $(z'_j, \mathbf{z}_{-j}) \in \mathcal{Z}$, $D_i(z_j, \mathbf{z}_{-j}) \geq D_i(z'_j, \mathbf{z}_{-j})$ when $z_j \geq_j z'_j$.*

Proof. To simplify notation take each ordering $\geq_j$ to be the ordering on the natural numbers $\geq$, without loss of generality. The two versions of VM are:

Assumption VM (vector monotonicity). *For $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$, if $\mathbf{z} \geq \mathbf{z}'$ component-wise, then $D_i(\mathbf{z}) \geq D_i(\mathbf{z}')$ for all i.*

Assumption VM' (alternative characterization). *$D_i(\mathbf{z}_j, \mathbf{z}_{-j}) \geq D_i(\mathbf{z}'_j, \mathbf{z}_{-j})$ for all i when $z_j \geq z'_j$ and both $(\mathbf{z}_j, \mathbf{z}_{-j})$ and $(\mathbf{z}'_j, \mathbf{z}_{-j}) \in \mathcal{Z}$*

The claim is that $VM \iff VM'$.

- $\mathbf{VM} \implies \mathbf{VM'}$: immediate, since $(\mathbf{z}_j, \mathbf{z}_{-j}) \geq (\mathbf{z}'_j, \mathbf{z}_{-j})$ in a vector sense when $z_j \geq z'_j$
- $\mathbf{VM'} \implies \mathbf{VM}$: consider $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$ such that $\mathbf{z} \geq \mathbf{z}'$ in a vector sense, i.e. $z_j \geq z'_j$ for all $j \in \{1 \dots J\}$. Since $\mathcal{Z}$ is non-disjoint, we can use By VM' to link $\mathbf{z}$ and $\mathbf{z}'$ by a series of single instrument changes (for some ordering of the instrument labels): $D_i(z_1, z_2, \dots z_J) \geq D_i(z'_i, z_2, \dots z_J)$, $D_i(z'_1, z_2, \dots z_J) \geq D_i(z'_i, z'_2, \dots z_J)$, and so on, for all i, and where all the intermediate instrument values are in $\mathcal{Z}$. Then, by transitivity of $\geq$ it follows that $D_i(\mathbf{z}) \geq D_i(\mathbf{z}')$ for all i.

□

Now we use the above lemma to establish Proposition 1. By the law of iterated expectations and Assumption 1:

$$P(\mathbf{z}) = \sum_{g \in \mathcal{G}} P(G_i = g | Z_i = \mathbf{z}) \cdot \mathbb{E}[D_i(Z_i) | G_i = g, Z_i = \mathbf{z}] = \sum_{g \in \mathcal{G}} P(G_i = g) \cdot D_g(\mathbf{z})$$

By VM, $D_g(\mathbf{z})$ is component-wise monotonic for any g in the support of G_i. As a convex combination of component-wise monotonic functions, $P(\mathbf{z})$ will also be component-wise monotonic.

In the other direction, note that by PM if $P(\mathbf{z}_j, \mathbf{z}_{-j}) > P(\mathbf{z}'_j, \mathbf{z}_{-j})$, then we must have that $D_i(\mathbf{z}_j, \mathbf{z}_{-j}) \geq D_i(\mathbf{z}'_j, \mathbf{z}_{-j})$ rather than $D_i(\mathbf{z}_j, \mathbf{z}_{-j}) \leq D_i(\mathbf{z}'_j, \mathbf{z}_{-j})$. Thus component-wise monotonicity of $P(\mathbf{z})$ with respect to some collection of orderings $\{\geq_j\}_{j \in \{1 \dots J\}}$ implies $D_i(\mathbf{z}_j, \mathbf{z}_{-j}) \geq D_i(\mathbf{z}'_j, \mathbf{z}_{-j})$ for all j, $z_j \geq_j z'_j$, and $\mathbf{z}_{-j} \in \mathcal{Z}_{-j}$. This verifies VM' from the Lemma above.

D.2 Proof of Proposition 2

For this proof, I'll use the notation $D_F(\cdot)$ for the $g \in \mathcal{G}^c$ for which $F = F(g)$, and $D_S(\cdot)$ for the $g \in \mathcal{G}^s$ for which $S = S(g)$.

Note that $\{D_F(\mathbf{z}) = 1\} \iff \left\{\bigcup_{S \in F} \{D_S(\mathbf{z}) = 1\}\right\} \iff$ not $\left\{\bigcap_{S \in F} \{D_S(\mathbf{z}) = 0\}\right\}$ for any fixed $\mathbf{z}$. The condition that $D_F(\mathbf{z}) = 1$ can thus be written as

$$D_{g(F)}(\mathbf{z}) = 1 - \prod_{S \in F} (1 - D_S(\mathbf{z})) = \sum_{\emptyset \subset F' \subseteq F} (-1)^{|F'|+1} \prod_{S \in F} D_S(\mathbf{z})$$

Let $\mathbf{z}_1(\mathbf{z}) = \{j \in \{1 \dots J\} : z_j = 1\}$ represent a vector of instrument values $\mathbf{z}$ as the subset of instrument indices for which the associated instrument takes the value of one. Then, using that for a simple response group $D_S(\mathbf{z}) = \mathbb{1}(S \subseteq \mathbf{z_1}(\mathbf{z}))$:

$$\begin{aligned}
D_{g(F)}(\mathbf{z}) &= \sum_{F' \subseteq F : F' \neq \emptyset} (-1)^{|F'|+1} \prod_{s \in F} D_S(\mathbf{z}) = \sum_{F' \subseteq F : F' \neq \emptyset} (-1)^{|F'|+1} \cdot D_{g\left(\left(\bigcup_{S \in F'} S\right)\right)}(\mathbf{z}) \\
&= \sum_{F' \subseteq F : F' \neq \emptyset} (-1)^{|F'|+1} \cdot \mathbb{1}\left(\left(\bigcup_{S \in F'} S\right) \subseteq \mathbf{z}(\mathbf{z})\right) = \sum_{\substack{\emptyset \subset F' \subseteq F: \\ \left(\bigcup_{S \in F'} S\right) \subseteq \mathbf{z}(\mathbf{z})}} (-1)^{|F'|+1} \\
&= \sum_{S' \subseteq \mathbf{z}(\mathbf{z})} \sum_{\substack{\emptyset \subset F' \subseteq F: \\ \left(\bigcup_{S \in F'} S\right) = S'}} (-1)^{|F'|+1} = \sum_{S' \subseteq \{1 \dots J\}} \mathbb{1}\left(S' \subseteq \mathbf{z}(\mathbf{z})\right) \sum_{\substack{\emptyset \subset F' \subseteq F: \\ \left(\bigcup_{S \in F'} S\right) = S'}} (-1)^{|F'|+1} \\
&= \sum_{S' \subseteq \{1 \dots J\}} \left[\sum_{\substack{\emptyset \subset F' \subseteq F: \\ \left(\bigcup_{S \in F'} S\right) = S'}} (-1)^{|F'|+1} \right] D_{S'}(\mathbf{z}) = \sum_{\emptyset \subset S' \subseteq \{1 \dots J\}} \left[\sum_{\substack{F' \subseteq F: \\ \left(\bigcup_{S \in F'} S\right) = S'}} (-1)^{|F'|+1} \right] D_{S'}(\mathbf{z})
\end{aligned}$$

Thus, letting $s(F, S') := \left\{F' \subseteq F : \left(\bigcup_{S \in F'} S\right) = S'\right\}$, we have $D_g(\mathbf{z}) = \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot D_{g'}(\mathbf{z})$ for any $g \in \mathcal{G}^c$ with $M_{gg'} := \sum_{F' \in s(F(g), S(g'))} (-1)^{|F'|+1}$.

D.3 Proof of Proposition 3

The if direction is most straightforward. From Proposition 2 we have that for any $\mathbf{z} \in \mathcal{Z}$ and $g \in \mathcal{G}^c$: $D_g(\mathbf{z}) = \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot D_{g'}(\mathbf{z})$. Thus, for any such $c(g, \mathbf{z})$:

$$
\begin{aligned}
c(g,\mathbf{z}) &= \sum_{k=1}^{K} \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot D_{g'}(\mathbf{u}_k(\mathbf{z}))) - \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot D_{g'}(\mathbf{l}_k(\mathbf{z}))) \\
&= \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot \left\{ \sum_{k=1}^{K} D_{g'}(\mathbf{u}_k(\mathbf{z}))) - D_{g'}(\mathbf{l}_k(\mathbf{z}))) \right\} \\
&= \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot c(g', \mathbf{z})
\end{aligned}
$$

for any $\mathbf{z} \in \mathcal{Z}$. To finish verifying Property M, we need only observe that $c(a.t., \mathbf{z}) = c(n.t., \mathbf{z}) = 0$ for all $\mathbf{z}$ since $D_g(\mathbf{u}_k(\mathbf{z})) = D_g(\mathbf{l}_k(\mathbf{z}))$ for any $\mathbf{u}_k, \mathbf{l}_k$ when $g \in \{a.t., n.t.\}$.

Now we turn to the other implication of the Proposition, that any c satisfying Property M has a representation like the above. For shorthand, let $c^{-1}(\mathbf{z})$ indicate the family of $S \subseteq \{1 \dots J\}$ such that $c(g(S), \mathbf{z}) = 1$. The following Lemma establishes that the family $c^{-1}(\mathbf{z})$ and its complement are each closed under unions:

Lemma. *Let c be a function from $\mathcal{G} \times \mathcal{Z}$ to $\{0, 1\}$ satisfies Property M. If $A \in c^{-1}(\mathbf{z})$ and $B \in c^{-1}(\mathbf{z})$, then $A \cup B \in c^{-1}(\mathbf{z})$, and if $A \notin c^{-1}(\mathbf{z})$ and $B \notin c^{-1}(\mathbf{z})$, then $A \cup B \notin c^{-1}(\mathbf{z})$.*

Proof. If the sets A and B are nested, then the result follows trivially. Now suppose neither set contains the other, and consider the Sperner family $A \sqcup B$ constructed of the two sets A and B. By Property M and using Proposition 2:

$$
\begin{aligned}
c(g(A \sqcup B), \mathbf{z}) &= \sum_{\emptyset \subset S' \subseteq \{1 \dots J\}} \left[\sum_{\substack{f \subseteq \{A,B\}: \\ (\bigcup_{S \in F'} S) = S'}} (-1)^{|F'|+1} \right] c\left(\bigcup_{S \in F'} S, z \right) = \sum_{\emptyset \subset F' \subseteq \{A,B\}} c\left(\bigcup_{S \in F'} S, z \right) \\
&= c(g(A), \mathbf{z}) + c(g(B), \mathbf{z}) - c(g(A \cup B), \mathbf{z})
\end{aligned}
$$

In the first case, if both A and B are in $c^{-1}(\mathbf{z})$, then we must have $c(g(A \cup B), \mathbf{z}) = 1$ to prevent $c(g(A \sqcup B), \mathbf{z})$ from evaluating to 2, which contradicts the assumption that c takes values in $\{0, 1\}$. In the second case, when both $c(g(A), \mathbf{z})$ and $c(g(B), \mathbf{z})$ are zero, we must have $c(g(A \cup B), \mathbf{z}) = 1$ to prevent $c(g(A \sqcup B), \mathbf{z})$ from evaluating to -1. □

As a consequence of the Lemma, since $c^{-1}(\mathbf{z})$ is a finite set, there exists a member $S_1(\mathbf{z})$ of $c^{-1}(\mathbf{z})$ that satisfies $S_1(\mathbf{z}) = \bigcup_{S \in c^{-1}(\mathbf{z})} S$ (similarly, there exists a $S_0(\mathbf{z}) = \bigcup_{S \notin c^{-1}(\mathbf{z})} S$ with $S_0(\mathbf{z}) \notin c^{-1}(\mathbf{z})$). All members of the family $c^{-1}(\mathbf{z})$ are subsets of $S_1(\mathbf{z})$, and all $S \subseteq \{1 \dots J\}$ that are not in $c^{-1}(\mathbf{z})$ are subsets of $S_0(\mathbf{z})$.

Let $\mathbf{z}$ take some fixed value, and beginning with the set $S_1 = S_1(\mathbf{z})$, define a sequence of sets $\{S_1, S_2, S_3, \dots\}$ as follows:

$$S_{2k} = \bigcup_{\substack{S' \subseteq S_{2k-1}: \\ S' \notin c^{-1}(\mathbf{z})}} S' \qquad \text{and} \qquad S_{2k+1} = \bigcup_{\substack{S' \subseteq S_{2k}: \\ S' \in c^{-1}(\mathbf{z})}} S'$$

where we take $\bigcup_{S' \in \emptyset} S'$ to evaluate to the empty set. This sequence provides a characterization of the family $c^{-1}(\mathbf{z})$ as follows. For any $\emptyset \subset S \subseteq \{1 \dots J\}$:

$$\begin{aligned}
c(g(S), \mathbf{z}) &= \mathbb{1}(S \in c^{-1}(\mathbf{z})) = \mathbb{1}(S \subseteq S_1 : S \in c^{-1}(\mathbf{z})) = \mathbb{1}(S \subseteq S_1) - \mathbb{1}(S \subseteq S_1 : S \notin c^{-1}(\mathbf{z})) \\
&= \mathbb{1}(S \subseteq S_1) - \big(\mathbb{1}(S \subseteq S_2) - \mathbb{1}(S \subseteq S_2 : S \in c^{-1}(\mathbf{z}))\big) \\
&= \mathbb{1}(S \subseteq S_1) - \mathbb{1}(S \subseteq S_2) + \big(\mathbb{1}(S \subseteq S_3) - \mathbb{1}(S \subseteq S_3 : S \notin c^{-1}(\mathbf{z}))\big) \\
&= \dots \\
&= \sum_{n=1}^{N} (-1)^{n+1} \cdot \mathbb{1}(S \subseteq S_n) + (-1)^N \cdot \begin{cases} \mathbb{1}(S \subseteq S_N : S \in c^{-1}(\mathbf{z})) & \text{if } N \text{ even} \\ \mathbb{1}(S \subseteq S_N : S \notin c^{-1}(\mathbf{z})) & \text{if } N \text{ odd} \end{cases}
\end{aligned}$$

for any natural number N.

Think of the power set of S_1 as a "first-order" approximation to the family $c^{-1}(\mathbf{z})$. However, in most cases this family is too large, as there will be subsets of S_1 that are not found in $c^{-1}(\mathbf{z})$. Define S_2 to be the union of all such offending sets. The power set of S_2 now provides a possible "overestimate" of the family of offending sets (since they are all in 2^{S_2}) and hence removing all subsets of S_2 as a correction to be applied to 2^{S_1} as an estimate of $c^{-2}(\mathbf{z})$ will overcompensate: we will have removed some sets which are indeed in $c^{-1}(\mathbf{z})$. We thus define S_3 analogously, whose power set provides an approximation to the error in S_2 as an approximation to the error in S_1, and so on.

Does this process of over-correction eventually terminate, so that the final remainder term is zero? Note that for any n: $S_n \subseteq S_{n-1}$. If $S_n = S_{n-1} \neq \emptyset$, then we have a fixed point S where $\bigcup_{S' \subseteq S : S' \in c^{-1}(\mathbf{z})} S' = \bigcup_{S' \subseteq S : S' \notin c^{-1}(\mathbf{z})} S'$. But by the Lemma, this would imply that S is a member both of $\{S' \subseteq S : S' \in c^{-1}(\mathbf{z})\}$ and of $\{S' \subseteq S : S' \notin c^{-1}(\mathbf{z})\}$, and therefore that both $c(g(S), \mathbf{z}) = 1$ and $c(g(S), \mathbf{z}) = 0$, a contradiction. Thus, $S_n \subset S_{n-1}$, and $|S_n|$ is a decreasing sequence of non-negative integers that is strictly decreasing so long as $|S_n| > 0$. It must thus converge to zero in at most $|S_1|$ iterations, so that $S_n = \emptyset$ for all $n \geq |S_1|$.

Without loss, we can terminate the sequence on an even term, since $\mathbb{1}(S \subseteq \emptyset) = 0$ for any $S \supset \emptyset$. Let $2K$ denote the smallest even number such that $S_n = \emptyset$ for all $n > 2K$, for a fixed $\mathbf{z}$. Thus, we have for any $\emptyset \subset S \subseteq \{1 \dots J\}$:

$$c(g(S), \mathbf{z}) = \sum_{n=1}^{2K} (-1)^{n+1} \cdot \mathbb{1}(S \subseteq S_n) = \sum_{k=1}^{K} \mathbb{1}(S \subseteq S_{2k-1}) - \mathbb{1}(S \subseteq S_{2k})$$

where $2K \leq |S_1| \leq J$. Recall that we have left the dependence of each of the sets S_n (as well as the integer K) on $\mathbf{z}$ implicit.

To obtain the notation of the final result, define for each k and $\mathbf{z}$ the point $\mathbf{u}_k(\mathbf{z}) \in \mathcal{Z}$ to have a value of one exactly for the elements in S_{2k-1} for that value of $\mathbf{z}$, and $\mathbf{l}_k(\mathbf{z}) \in \mathcal{Z}$ to have a value of one exactly for the elements in S_{2k} for that value of $\mathbf{z}$. We may thus write, for any $g \in \mathcal{G}^s$ (corresponding to a $\emptyset \subset S(g) \subseteq \{1 \dots J\}$) and any $\mathbf{z} \in \mathcal{Z}$:

$$c(g, \mathbf{z}) = \sum_{k=1}^{K(\mathbf{z})} D_g(\mathbf{u}_k(\mathbf{z})) - D_g(\mathbf{l}_k(\mathbf{z})) = \sum_{k=1}^{K} D_g(\mathbf{u}_k(\mathbf{z})) - D_g(\mathbf{l}_k(\mathbf{z}))$$

where we let K be the maximum of $K(\mathbf{z})$ over the finite set $\mathcal{Z}$, and we define $\mathbf{u}_k(\mathbf{z})$ and $\mathbf{l}_k(\mathbf{z})$ to each be a vector of zeros whenever $k > K(\mathbf{z})$. For each $\mathbf{z}$, the relations $\mathbf{u}_k(\mathbf{z}) \geq \mathbf{l}_k(\mathbf{z})$ and $\mathbf{l}_k(\mathbf{z}) \geq \mathbf{u}_{k+1}(\mathbf{z})$ component-wise now follow from $S_n \subseteq S_{n+1}$.

Now we may apply Property M to construct $c(g, \mathbf{z})$ for any of the non-simple response groups as well. Recall that Property M says that $c(g, \mathbf{z}) = \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot c(g', \mathbf{z})$ for all $\mathbf{z}$, and any $g \in \mathcal{G}^c$. Thus:

$$\begin{aligned}
c(g, \mathbf{z}) &= \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot \sum_{k=1}^{K} \{D_{g'}(\mathbf{u}_k(\mathbf{z})) - D_{g'}(\mathbf{l}_k(\mathbf{z}))\} \\
&= \sum_{k=1}^{K} \left\{ \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot D_{g'}(\mathbf{u}_k(\mathbf{z})) \right\} - \left\{ \sum_{g' \in \mathcal{G}^s} M_{gg'} \cdot D_{g'}(\mathbf{l}_k(\mathbf{z})) \right\} \\
&= \sum_{k=1}^{K} D_g(\mathbf{u}_k(\mathbf{z})) - D_g(\mathbf{l}_k(\mathbf{z}))
\end{aligned}$$

Finally, note that $D_g(\mathbf{u}_k(\mathbf{z})) = D_g(\mathbf{l}_k(\mathbf{z}))$ for any $g \in \{a.t., n.t.\}$ so the above expression $c(g, \mathbf{z}) = \sum_{k=1}^{K} D_g(\mathbf{u}_k(\mathbf{z})) - D_g(\mathbf{l}_k(\mathbf{z}))$ holds for all $g \in \mathcal{G}$.

D.4 Proof of Proposition 4

Let $\tilde{\mathcal{Z}}$ be the set of possible values for the new set of instruments $(\tilde{Z}_2, \dots \tilde{Z}_m, Z_{-1})$, where Z_{-1} is a shorthand for $(Z_2, \dots, Z_J)$. Note that fixing the value of Z_1 is equivalent to fixing the values of all of $\tilde{Z}_1 \dots \tilde{Z}_M$. Since $P(\tilde{Z}_{mi} = 0 \,\&\, \tilde{Z}_{ni} = 1) = 0$ for any $m > n$, we may without loss take $\tilde{\mathcal{Z}}$ to consist only of cases where $\tilde{Z}_1 \dots \tilde{Z}_M$ takes the form $(\underbrace{0, \dots 0}_{m-1}, \underbrace{1, \dots 1}_{M-m+1})$ for some m. Let $\tilde{Z}_{-m}$ denote all of the instruments in $\tilde{Z}_1 \dots \tilde{Z}_M$ aside from $\tilde{Z}_m$.

If $\mathcal{Z}$ is non-disjoint, then the $\tilde{\mathcal{Z}}$ given above is also non-disjoint. Then, by Proposition D.1, we simply need to show that $D_i(1, \tilde{\mathbf{z}}_{-m}; \mathbf{z}_{-1}) \geq D_i(0, \tilde{\mathbf{z}}_{-m}; \mathbf{z}_{-1})$ for any $\mathbf{z}_{-1}$ and $\tilde{\mathbf{z}}_{-m}$ such that $(0, \tilde{\mathbf{z}}_{-m}, \mathbf{z}_{-1}), (1, \tilde{\mathbf{z}}_{-m}, \mathbf{z}_{-1}) \in \mathcal{Z}$, where the notation $D_i(a, b; c)$ is understood as $D_i(d, c)$ where d is the value of Z_1 corresponding to $\tilde{Z}$ with value a for $\tilde{Z}_m$ and b for $\tilde{Z}_{-m}$. For any $\tilde{\mathbf{z}}_{-m}$ satisfying $(0, \tilde{\mathbf{z}}_{-m}, \mathbf{z}_{-1}) \in \mathcal{Z}$ and $(1, \tilde{\mathbf{z}}_{-m}, \mathbf{z}_{-1}) \in \mathcal{Z}$, switching $\tilde{Z}_m$ from zero to ones corresponds to switching instrument Z_1 from value z_{m-1} to value z_m. Since $D_i(1, \tilde{\mathbf{z}}_{-m}; \mathbf{z}_{-1}) - D_i(0, \tilde{\mathbf{z}}_{-m}; \mathbf{z}_{-1}) = D_i(z_m, \mathbf{z}_{-1}) - D_i(z_{m-1}, \mathbf{z}_{-1}) \geq 0$ by vector monotonicity on the original vector $(Z_1 \dots Z_J)$, the result follows.

D.5 Proof of Proposition 5

Introduce the notation that $\sqcup$ indicates taking union of two families of sets, e.g. adding a new set to the family. Let $\mathcal{T} := \{\tilde{Z}^j_m\}_{\substack{j\in\{1\dots J\}\\ m=1\dots M_j}}$ be the collection defined in Proposition 5. Consider any $S \subseteq \mathcal{T}$ that contains both Z^j_m and $Z^j_{m'}$ for some j and $m < m'$, and any other Sperner family F. For any Sperner family F, the families $g(F \sqcup S)$ and $g(F \sqcup (S - \{\tilde{Z}^j_m\}))$ generate the same selection behavior on all of $\mathcal{Z}$, because $\tilde{Z}^j_{m'} = 1 \implies \tilde{Z}^j_m = 1$. Therefore, if we let $\mathcal{F}$ be the family of all $S \subset \mathcal{T}$ that contain either no $\tilde{Z}^j_m$ for any given j or all $\tilde{Z}^j_{m_j}$ up to some m_j, this choice of $\mathcal{F}$ satisfies Assumption 3b*.

Note that given the assumption that $\mathbb{S}_Z = (\mathcal{Z}_1 \times \mathcal{Z}_2 \times \cdots \times \mathcal{Z}_J)$, the set of values $\mathcal{Z}$ that the original instruments can take must also be rectangular, and $\mathcal{F}$ is then isomorphic to $\mathcal{Z}$. We can construct this isomorphism between a $\mathbf{z} = (z_1, \dots z_J)' \in \mathcal{Z}$ explicitly as follows. Let us without loss of generality label the values of instrument Z_j by the consecutive non-negative integers $0 \dots M_j$. Any j for which there is no $\tilde{Z}^j_m \in S$ for some m is associated with $z_j = 0$, while for any other j the largest m for which $\tilde{Z}^j_m \in S$ we associate with $z_j = m$.

Now we are ready to show that the above choice of $\mathcal{F}$ satisfies Assumption 3a*. Suppose that it did not, i.e. there existed a non-zero vector ω such that $P\left(\sum_{S\in\mathcal{F}} \omega_S Z_{Si} = 0\right) = 1$ with $Z_{Si} := \prod_{Z^j_m \in S} \tilde{Z}^j_m$. This would imply non-invertibility of $\mathbb{E}[(1, \Gamma_i')(1, \Gamma_i')']$, where $\Gamma_i := \{Z_{Si}\}_{S\in\mathcal{F}, S\neq\emptyset}$. Note that $\Sigma := Var(\Gamma_i)$ has full rank iff $\omega'\mathbb{E}[(\Gamma_i - \mathbb{E}[\Gamma_i])(\Gamma_i - \mathbb{E}[\Gamma_i])]\omega = \mathbb{E}[\omega'(\Gamma_i - \mathbb{E}[\Gamma_i])(\Gamma_i - \mathbb{E}[\Gamma_i])\omega] > 0$, i.e. $P(\omega'(\Gamma_i - \mathbb{E}[\Gamma_i]) = 0) < 1$, for any non-zero $\omega \in \mathbb{R}^k$, where $k = |\mathcal{F}| - 1$. Similarly Σ^* has full rank if for any $\omega_0 \in \mathbb{R}, \omega \in \mathbb{R}^k$, $P((\omega_0, \omega)'(1, \Gamma_i) = 0) < 1$, where (ω_0, ω) is not the zero vector in $\mathbb{R}^{k+1}$. But if for some ω, $\omega'(\Gamma_i - \mathbb{E}[\Gamma_i]) = 0$ w.p.1, then we also have $(\omega_0, \omega)'(1, \Gamma_i) = 0$ w.p.1. by choosing $\omega_0 = -\omega'\mathbb{E}[\Gamma_i]$. In the other direction, note that $(\omega_0, \omega)'(1, \Gamma_i) = 0$ w.p.1. implies that $\omega'\Gamma_i = -\omega_0$ and hence $\omega'(\Gamma_i - \mathbb{E}[\Gamma_i]) = -\omega_0 - \omega'\mathbb{E}[\Gamma_i] = -\omega_0 - \mathbb{E}[\omega'\Gamma_i] = -\omega_0 + \omega_0 = 0$. Thus Σ has full rank of $k+1$ if and only if Σ^* has full rank of k, which occurs if and only if $P(\tilde{\omega}'(1, \Gamma_i) = 0) < 1$, where $\tilde{\omega}$ is any non-zero vector in $\mathbb{R}^{|\mathcal{F}|}$. In the case that $\mathcal{F}$ contains the family of all 2^J subsets of the instruments, we've now shown that Σ is invertible iff Assumption 3 holds (Lemma 1).

The remainder of this proof considers the general case and shows that if the rank of Σ were less than $|\mathcal{Z}|$, it would require the support of the original instruments to be non-rectangular, i.e. $|\mathbb{S}_{orig}| < \mathcal{Z}$ (where recall that $\mathcal{Z}$ denotes the set of values the original discrete instruments, rather than the generated binary instruments, can take). Observe that for any two $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$:

$$
\begin{aligned}
\mathbb{1}(\mathbf{z} = \mathbf{z}') &= \sum_{\mathbf{m} \in \mathcal{Z}} \mathbb{1}(\mathbf{z} = \mathbf{m}) \cdot \mathbb{1}(\mathbf{z}' = \mathbf{m}) \\
&= \sum_{\mathbf{m} \in \mathcal{Z}} \prod_{j=1}^{J} \mathbb{1}(z_j = m_j) \cdot \prod_{j=1}^{J} \mathbb{1}(z'_j = m_j) \\
&= \sum_{\mathbf{m} \in \mathcal{Z}} \left(\underbrace{\prod_{j=1}^{J} \{\mathbb{1}(z_j \geq m_j) - \mathbb{1}(z_j \geq m_j + 1)\}}_{\mathcal{M}_{\mathbf{z},\mathbf{m}}} \right) \cdot \left(\underbrace{\prod_{j=1}^{J} \{\mathbb{1}(z'_j \geq m_j) - \mathbb{1}(z'_j \geq m_j + 1)\}}_{\mathcal{M}_{\mathbf{z}',\mathbf{m}}} \right)
\end{aligned} \tag{1}
$$

where we define an $|\mathcal{Z}| \times |\mathcal{Z}|$ matrix $\mathcal{M}$ with entries given as above. The above equation shows that $\mathcal{M}^{-1}$ exists and is equal to $\mathcal{M}'$, because (1) can be written as $I_{\mathbf{z},\mathbf{z}'} = [\mathcal{M}\mathcal{M}']_{\mathbf{z},\mathbf{z}'}$, where I is the identity matrix on $\mathbb{R}^{|\mathcal{Z}|}$.

Note that we can write the entries of $\mathcal{M}$ as:

$$
\begin{aligned}
\mathcal{M}_{\mathbf{z},\mathbf{m}} = \prod_{j=1}^{J} \{\mathbb{1}(z_j \geq m_j) - \mathbb{1}(z_j \geq m_j + 1)\} &= \sum_{S \subseteq \{1 \ldots J\}} (-1)^{|S|} \cdot \prod_{j \in S} \mathbb{1}(z_j \geq m_j + 1) \cdot \prod_{j \notin S} \mathbb{1}(z_j \geq m_j) \\
&= \sum_{\mathbf{m}' \in \mathcal{Z}} \tilde{D}_{\mathbf{z},\mathbf{m}'} \tilde{\mathcal{A}}_{\mathbf{m}',\mathbf{m}}
\end{aligned} \tag{2}
$$

where $\tilde{D}_{\mathbf{z},\mathbf{m}} := \prod_{j=1}^{J} \mathbb{1}(z_j \geq m_j)$, and

$$
\tilde{\mathcal{A}}_{\mathbf{m}',\mathbf{m}} := \begin{cases} (-1)^{|S|} & \text{if there exists an } S \subseteq \{1 \ldots J\} : m'_j = \begin{cases} m_j + 1 & \text{if } j \in S \\ m_j & \text{if } j \notin S \end{cases} \\ 0 & \text{otherwise} \end{cases}
$$

and note that the set S is unique when $\tilde{\mathcal{A}}_{\mathbf{m},\mathbf{m}'} \neq 0$. Thus, $\mathcal{M} = \tilde{D}\tilde{\mathcal{A}}$, and since $\mathcal{M}$, $\tilde{\mathcal{A}}$ and $\tilde{D}$ are all square $|\mathcal{Z}| \times |\mathcal{Z}|$ matrices, invertibility of $\mathcal{M}$ implies that both $\tilde{\mathcal{A}}$ and $\tilde{D}$ are invertible. The special case of this equality in the baseline setup of the main paper, in which the instruments are binary and $\mathbb{S}_Z = \mathcal{Z} = \{0,1\}^J$, is discussed in detail in subsection D.5.1.

For any $\mathbf{m} \in \mathcal{Z}$, let $Z_{\mathbf{m},i} = \prod_{j=1}^{J} \mathbb{1}(Z_{ji} \geq m_j)$. Let $(1, \Gamma'_i)'_i$ be a vector of $Z_{\mathbf{m},i}$ across all $\mathbf{m} \in \mathcal{Z}$ where we separate out the first element corresponding to $\mathbf{m} = (0, \ldots 0)'$ (generalizing the notation of Lemma 1, where the first element was associated with $S = \emptyset$). With probability one: $[(1, \Gamma'_i)']_{\mathbf{m}} = \tilde{D}_{Z_i,\mathbf{m}} = \sum_{\mathbf{z} \in \mathcal{Z}} [\mathfrak{Z}_i]_{\mathbf{z}} \cdot \tilde{D}_{\mathbf{z},\mathbf{m}} = [\tilde{D}'\mathfrak{Z}_i]_{\mathbf{m}}$, where $\mathfrak{Z}_i$ is a vector with entries $[\mathfrak{Z}_i]_{\mathbf{z}} = \mathbb{1}(Z_i = \mathbf{z})$. Therefore, $\Sigma^* := \mathbb{E}[(1, \Gamma'_i)'(1, \Gamma'_i)] = \tilde{D}'\mathbb{E}[\mathfrak{Z}_i\mathfrak{Z}'_i]\tilde{D} = \tilde{D}'P\tilde{D}$, where P is a diagonal $|\mathcal{Z}| \times |\mathcal{Z}|$ matrix with entries $P_{\mathbf{z},\mathbf{z}} = P(Z_i = \mathbf{z})$ for each $\mathbf{z} \in \mathcal{Z}$. Note that $\mathbb{E}[\mathfrak{Z}_i\mathfrak{Z}'_i]$ is diagonal because events in which Z_i takes on different values $\mathbf{z}$ are exclusive:

$\mathbb{E}[\mathbb{1}(Z_i = \mathbf{z})\mathbb{1}(Z_i = \mathbf{z}')] = 0$ for any $\mathbf{z} \neq \mathbf{z}'$. Since $\tilde{D}^{-1}$ exists, the rank of $\Sigma^* = \tilde{D}P\tilde{D}'$ is equal to the rank of P, which is $|\mathbb{S}_{orig}|$. I have shown above that if Assumption 3a* did not hold, then $rank(\Sigma^*) = |\mathbb{S}_Z|$ would need to be less than $|\mathcal{Z}|$. We can thus conclude that if Assumption $3a^*$ is violated (for the choice of $\mathcal{F}$ described in Proposition 5), $|\mathbb{S}_Z| < |\mathcal{Z}|$ and we cannot have full rectangular support for the original instruments.

D.5.1 Special case with binary instruments

In the special case of binary instruments, $M_j = 1$ and the rows of $\tilde{\mathcal{A}}$ and columns of $\tilde{D}$ can be indexed by subsets S of $\{1 \dots J\}$ (corresponding to the values of $\mathbf{m}$ for which $m_j = 1$). Let $\mathbf{z}(S)$ be the unique J component binary vector such that $\mathbf{z}_1 = S$ (where $\mathbf{z}_1$ indicates the set of values for which $z_j = 1$). The entries of $\tilde{\mathcal{A}}$ are $\tilde{\mathcal{A}}_{S,\mathbf{z}} = (-1)^{|S-\mathbf{z}_1|} \cdot \mathbb{1}(\mathbf{z}_1 \subseteq S)$. The entries of $\mathcal{M}$ can then be written, using (2), as:

$$\mathcal{M}_{\mathbf{z},\mathbf{z}'} = \sum_{S:\mathbf{z}(S)\in\mathcal{Z}} \tilde{D}_{\mathbf{z},S}\tilde{\mathcal{A}}_{S,\mathbf{z}'} = \sum_{S:\mathbf{z}(S)\in\mathcal{Z}} (-1)^{|S-\mathbf{z}_1'|} \cdot \mathbb{1}(\mathbf{z}_1' \subseteq S) \cdot \mathbb{1}(S \subseteq \mathbf{z}_1) = \sum_{\substack{S:\mathbf{z}(S)\in\mathcal{Z}: \\ \mathbf{z}_1' \subseteq S \subseteq \mathbf{z}_1}} (-1)^{|S-\mathbf{z}_1'|}$$

When $\mathcal{Z} = \{0,1\}^J$ so that $\{S : \mathbf{z}(S) \in \mathcal{Z} \text{ and } \mathbf{z}_1' \subseteq S \subseteq \mathbf{z}_1\} = \{S \subseteq \{1 \dots J\} : \mathbf{z}_1' \subseteq S \subseteq \mathbf{z}_1\}$, $\mathcal{M}_{\mathbf{z},\mathbf{z}'} = \sum_{\mathbf{z}_1' \subseteq S \subseteq \mathbf{z}_1} (-1)^{|S-\mathbf{z}_1'|} = \mathbb{1}(\mathbf{z}_1' = \mathbf{z}_1) = \mathbb{1}(\mathbf{z}' = \mathbf{z})$, so $\mathcal{M}$ becomes the identity matrix $I_{|\mathcal{Z}|}$ on $\mathbb{R}^{|\mathcal{Z}|}$, i.e. $\tilde{D}\tilde{\mathcal{A}} = I_{|\mathcal{Z}|}$. Since $\tilde{\mathcal{A}}$ and $\tilde{D}$ are each $|\mathcal{Z}| \times |\mathcal{Z}|$ square matrices, we have that $\tilde{D}^{-1}$ exists and is equal to $\tilde{\mathcal{A}}$ when $\mathcal{Z} = \{0,1\}^J$.

Proof of Lemma 2 when Assumption 3* replaces Assumption 3

Let $\mathbb{P}_D = D(D'D)^{-1}D'$ be an orthogonal projection matrix into the column-space of D, and $\mathbb{P}_{P^{1/2}D} = P^{1/2}D(D'PD)^{-1}D'P^{1/2}$ an orthogonal projection matrix into the column-space of $P^{1/2}D$. These expressions for the projections follow because D has full column-rank of $|\mathcal{F}|$ under Assumption 3* (see proof of Proposition 6 for a demonstration of this fact), and therefore $P^{1/2}D$ does as well since P is invertible.

I will prove the result of the Lemma by first showing that for any vector $\mathbf{w} \in \mathbb{R}^{|\mathcal{S}|}$, $P^{1/2}\mathbb{P}_D\mathbf{w} = \mathbb{P}_{P^{1/2}D}P^{1/2}\mathbf{w}$, i.e. transforming $\mathbf{w}$ by $P^{1/2}$ before projecting onto the column-space of $P^{1/2}D$ is equivalent to projecting the un-transformed vector $\mathbf{w}$ onto the column-space of D, and then transforming it by $P^{1/2}$. This implies that $\mathbb{P}_D = P^{-1/2}\mathbb{P}_{P^{1/2}D}P^{1/2}$. Pre-multiplying this equation by $(D'D)^{-1}D'$, we then obtain

$$\cancel{(D'D)^{-1}D'D}(D'D)^{-1}D' = \cancel{(D'D)^{-1}D'P^{-1/2}}P^{1/2}D(D'PD)^{-1}D'P^{1/2}P^{1/2},$$

or $(D'D)^{-1}D' = (D'PD)^{-1}D'$. The result then follows using that $D^+ = (D'D)^{-1}D'$ because D has full column-rank.

To see that $P^{1/2}\mathbb{P}_D\mathbf{w} = \mathbb{P}_{P^{1/2}D}P^{1/2}\mathbf{w}$ for any $\mathbf{w} \in \mathbb{R}^{\mathbb{S}_Z}$, note that we can decompose $\mathbb{R}^{\mathbb{S}_Z}$ as $Range(D) \oplus Null(D')$, or alternatively as $Range(PD) \oplus Null(D'P)$. As a result, we can write any $\mathbf{w} \in \mathbb{R}^{|\mathbb{S}_Z|}$ as $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 + \mathbf{w}_4$, where $\mathbf{w}_1 \in Range(PD) \cap Range(D)$, $\mathbf{w}_2 \in Null(D'P) \cap Range(D)$, $\mathbf{w}_3 \in Null(D'P) \cap Null(D')$, and $\mathbf{w}_4 \in Range(PD) \cap Null(D')$. Note that $\mathbf{x} \in Range(D)$ means that $\mathbf{x} = D\alpha$ for some $\alpha \in \mathbb{R}^{|\mathcal{F}|}$.

Consider first the component $\mathbf{w}_4 \in Range(PD) \cap Null(D')$. Since $\mathbf{w}_4 \in Range(PD)$, $\mathbf{w}_4 = PD\alpha$ for some α. Meanwhile, since $\mathbf{w}_4 \in Null(D')$, we have that $D'\mathbf{w}_4 = \mathbf{0}$. Together, this implies that $D'PD\alpha = \mathbf{0}$, which implies that $\alpha = 0$ given that $(D'PD)$ is invertible (it must be given that D is rank $|\mathcal{F}|$, P is rank $|\mathbb{S}_Z| \geq |\mathcal{F}|$, and $(D'PD)$ is $|\mathcal{F}| \times |\mathcal{F}|$). Therefore we can conclude that $\mathbf{w}_4 = \mathbf{0}$, and our goal becomes only to show that $P^{1/2}\mathbb{P}_D(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3) = \mathbb{P}_{P^{1/2}D}P^{1/2}(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3)$.

Consider next the sum of $\mathbf{w}_1$ and $\mathbf{w}_2$ only. Since $\mathbf{w}_1 + \mathbf{w}_2 \in Range(D)$ implies that we can write $\mathbf{w}_1 + \mathbf{w}_2 = D\alpha$, we have that $\mathbb{P}_{P^{1/2}D}P^{1/2}(\mathbf{w}_1 + \mathbf{w}_2) = P^{1/2}D\cancel{(D'PD)^1 D'PD}\alpha = P^{1/2}D\alpha = P^{1/2}(\mathbf{w}_1 + \mathbf{w}2)$, while $P^{1/2}\mathbb{P}_D(\mathbf{w}_1 + \mathbf{w}_3) = P^{1/2}(\mathbf{w}_1 + \mathbf{w}_2)$ as well.

Finally, consider $\mathbf{w}_3$. That $\mathbf{w}_3 \in Null(D'P)$ implies that $D'P\mathbf{w}_3 = \mathbf{0}$. Therefore $\mathbb{P}_{P^{1/2}D}P^{1/2}\mathbf{w}_3 = P^{1/2}D(D'PD)^1 D'P\mathbf{w}_3 = \mathbf{0}$. Meanwhile, that $\mathbf{w}_3 \in Null(D')$ implies that $D'\mathbf{w}_3 = \mathbf{0}$. Therefore $P^{1/2}\mathbb{P}_D\mathbf{w}_3 = P^{1/2}D(D'D)^{-1}D'\mathbf{w}_3 = \mathbf{0}$ as well.

Proof of Lemma 4 when Assumption 3* replaces Assumption 3

I begin by showing that while D is not necessarily square when Assumption 3 is relaxed to Assumption 3a* (and hence we can no longer use that D^{-1} exists and equals $\mathcal{A}$), it is still the case that D has full column rank. This implies that $A(d)^+ = \left(D\tilde{M}(d)\right)^+ = \tilde{M}(d)'^+ D^+$.

To see that D has full column-rank under Assumption 3*, recall that the row and column ranks of any matrix are equal. Thus it suffices for this to show that the $|\mathbb{S}_Z| \times |\mathcal{F}|$ matrix D' has full row rank of $|\mathcal{F}|$. If D' has $|\mathcal{F}|$ linearly independent rows, then it also has $|\mathcal{F}|$ linearly independent columns, and thus a column space that spans all of $\mathbb{R}^{|\mathcal{F}|}$. Suppose instead that the $|\mathcal{F}|$ rows of D' were not linearly independent. Then there would exist a $\omega \in \mathbb{R}^{|\mathcal{F}|}/\mathbf{0}$ such that $\sum_{S \in \mathcal{F}} \omega_{\mathbf{S}} \cdot D'_{S,\mathbf{z}} = 0$ for all $\mathbf{z} \in \mathbb{S}_Z$. Note that for any $S \in \mathcal{F}$: $Z_{Si} = D_S(Z_i) = D'_{S,Z_i} = \sum_{\mathbf{z} \in \mathbb{S}_Z} \mathbb{1}(Z_i = \mathbf{z}) \cdot D'_{S,\mathbf{z}}$ with probability one. Thus, if such an ω existed, we would have

$$\sum_{S \in \mathcal{F}} \omega_S \cdot Z_{S,i} = \sum_{S \in \mathcal{F}} \omega_S \sum_{\mathbf{z} \in \mathbb{S}_Z} \mathbb{1}(Z_i = \mathbf{z}) \cdot D'_{S,\mathbf{z}} = \sum_{\mathbf{z} \in \mathbb{S}_Z} \mathbb{1}(Z_i = \mathbf{z}) \left(\sum_{S \in \mathcal{F}} \omega_S \cdot D'_{S,\mathbf{z}} \right) = \sum_{\mathbf{z} \in \mathbb{S}_Z} \mathbb{1}(Z_i = \mathbf{z}) \cdot 0$$

which is equal to zero with probability one, violating Assumption 3a*.

Now, using Eq. (7) and that $A(d)^+ = \tilde{M}(d)'^+ D^+$:

$$\theta' A(d)^+ A(d) = \frac{(-1)^{d+1}}{\mathbb{E}[c(G_i, Z_i)]} \tilde{\lambda}' \tilde{M}(d)' A(d)^+ A(d) = \frac{(-1)^{d+1}}{\mathbb{E}[c(G_i, Z_i)]} \tilde{\lambda}' \cancel{\tilde{M}(d)' \tilde{M}(d)'^+} \cancel{D^+ D} \tilde{M}(d)' = \theta'$$

We can cross out $\tilde{M}(d)'\tilde{M}(d)'^+ = (\tilde{M}(d)^+ \tilde{M}(d))' = (I_{|\mathcal{F}|})' = I_{|\mathcal{F}|}$ by Lemma 3, and the above result that D has full column-rank and hence $D^+ D = I_{|\mathcal{F}|}$ to cross out $D^+ D$.

Direct proof of Theorem 1*

Note that the function $h(\cdot)$ given in Theorem 1 has the property that $\mathbb{E}[h(Z_i)] = 0$, for any distribution of the instruments. Consider the quantity $\mathbb{E}[Y_i D_i h(Z_i)]$ for a function h having this property. By the law of iterated expectations, and Assumption 1

$$\mathbb{E}[Y_i D_i h(Z_i)] = \sum_{g \in \mathcal{G}} P(G_i = g) \cdot \mathbb{E}[Y_i(1)|G_i = g] \cdot \mathbb{E}[D_g(Z_i) \cdot h(Z_i)] \tag{3}$$

Similarly

$$\begin{aligned}\mathbb{E}[Y_i(1-D_i)h(Z_i)] &= \sum_{g \in \mathcal{G}} P(G_i = g) \cdot \mathbb{E}[Y_i(0)(1-D_i)h(Z_i)|G_i = g] \\ &= -\sum_{g \in \mathcal{G}} P(G_i = g) \cdot \mathbb{E}[Y_i(0)|G_i = g] \cdot \mathbb{E}[D_g(Z_i) \cdot h(Z_i)] \end{aligned} \tag{4}$$

where I have used that $Z_i \perp\!\!\!\perp (Y_i(0), Z_i)$ and $\mathbb{E}[h(Z_i)] = 0$. Note that in Equations (3) and (4), the weighing over various groups g is governed by the quantity $\mathbb{E}[D_g(Z_i)h(Z_i)]$. It can be seen that never takers and always takers receive no weight, since $\mathbb{E}[D_{n.t}(Z_i)h(Z_i)] = \mathbb{E}[0] = 0$ and since $\mathbb{E}[D_{a.t}(Z_i)h(Z_i)] = \mathbb{E}[h(Z_i)] = 0$.

Similarly, for a causal parameter of the form $\mu_c^d = \mathbb{E}[Y_i(d)|C_i = 1]$, we can use the law of iterated expectations and Assumption 1 to write:

$$\begin{aligned}\mu_c^d &= \sum_{g \in \mathcal{G}} P(G_i = g|C_i = 1) \cdot \mathbb{E}[Y_i(d)|G_i = g, c(g, Z_i) = 1] \\ &= P(C_i = 1)^{-1} \cdot \sum_{g \in \mathcal{G}} P(C_i = 1|G_i = g) \cdot P(G_i = g) \cdot \mathbb{E}[Y_i(d)|G_i = g] \\ &= P(C_i = 1)^{-1} \cdot \sum_{g \in \mathcal{G}} \mathbb{E}[c(g, Z_i)] \cdot P(G_i = g) \cdot \mathbb{E}[Y_i(d)|G_i = g] \\ &= P(C_i = 1)^{-1} \cdot \sum_{g \in \mathcal{G}^c} \left\{ \sum_{g' \in \mathcal{G}^s} M_{g,g} \cdot \mathbb{E}[c(g(S), Z_i)] \right\} \cdot P(G_i = g) \cdot \mathbb{E}[Y_i(d)|G_i = g] \end{aligned} \tag{5}$$

where the final equality uses Property M. If Assumption 3 holds, recall that $\mathcal{G}^s$ is composed of simple compliance groups $g(S)$ for all non-null subsets $S \subseteq \{1 \dots J\}$. More generally, we can define $\mathcal{G}^s = \{g(S) : S \in \mathcal{F}, S \neq \emptyset\}$ where $\mathcal{F}$ is any collection of sets S that satisfies Assumption 3*. When Assumption 3 holds, $\mathcal{F}$ is the full powerset $2^{\{1 \dots J\}}$ of $\{1 \dots J\}$.

By Assumption 3b*, $\mathcal{F}$ is sufficiently rich to expand the selection functions $D_g(\cdot)$ for any g in $\mathcal{G}^c = \mathcal{G}/\{a.t., n.t.\}$ over the $g' \in \mathcal{G}^s$ corresponding to non-empty simple compliance groups $S \in \mathcal{F}$, i.e. $D_g(\cdot) = \sum_{S \in \mathcal{F}, S \neq \emptyset} M_{g,g(S)} D_{g(S)}(\cdot)$. Thus we can rewrite Equations (3) and (4) by substituting $\mathbb{E}[D_g(Z_i) \cdot h(Z_i)] = \sum_{S \in \mathcal{F}, S \neq \emptyset} M_{g,g(S)} \cdot \mathbb{E}[D_{g(S)}(Z_i) \cdot h(Z_i)]$, where again the sum is over the columns of the matrix M that correspond to non-empty $S \in \mathcal{F}$. Comparing to Eq. (5), we can match the coefficients appearing on each $P(G_i = g) \cdot \mathbb{E}[Y_i(d)|G_i = g]$ in

μ_c^d with those that would appear in $P(C_i = 1) \cdot \mathbb{E}[Y_i \mathbb{1}(D_i = d) h(Z_i)]$, if we can choose $h(\cdot)$ in such a way that $\mathbb{E}[D_{g(S)}(Z_i) \cdot h(Z_i)] = \mathbb{E}[c(g(S), Z_i)]$, for all $S \in \mathcal{F}$.

I now show that this is possible given Assumption 3a*. Assumption 3a* implies that Σ^{-1} exists (see proof of Proposition 5), where $\Sigma = Var(\Gamma_i)$ and Γ_i is a vector of $D_g(Z_i)$ over all $g \in \mathcal{G}^s$ (equivalently, a vector of $Z_{S,i} := \prod_{j \in S} Z_{j,i}$ over all $S \in \mathcal{F}, S \neq \emptyset$). In particular, with the choice $h(Z_i) = (\Gamma_i - \mathbb{E}[\Gamma_i])'\Sigma^{-1}\lambda$ for some vector $\lambda \in \mathbb{R}^k$ where $k = |\mathcal{G}^s| = |\mathcal{F}| - 1$:

$$\begin{aligned}(\mathbb{E}[h(Z_i), \Gamma_{1i}], \mathbb{E}[h(Z_i), \Gamma_{2i}], \dots, \mathbb{E}[h(Z_i), \Gamma_{ki}])' &= \mathbb{E}[(\Gamma_i - \mathbb{E}[\Gamma_i])h(Z_i)] \\ &= \mathbb{E}[(\Gamma_i - \mathbb{E}[\Gamma_i])(\Gamma_i - \mathbb{E}[\Gamma_i])']\Sigma^{-1}\lambda = \Sigma\Sigma^{-1}\lambda = \lambda \qquad (6)\end{aligned}$$

where in the first step I have used that $\mathbb{E}[h(Z_i)] = 0$. Note that that $\lambda_{g'} = \mathbb{E}[c(g', Z_i)]$ can be computed for each $g' \in \mathcal{G}^s$ from the observed distribution of Z and knowledge of the function c. Finally, to see that $P(C_i = 1) = \mathbb{E}[h(Z_i)D_i]$ we can use Assumption 1 and Property M:

$$\begin{aligned}\mathbb{E}[h(Z_i)D_i] &= \sum_{g \in \mathcal{G}} P(G_i = g)\mathbb{E}[h(Z_i)D_g(Z_i)] = \sum_{g \in \mathcal{G}^c} P(G_i = g)\mathbb{E}[h(Z_i)D_g(Z_i)] \\ &= \sum_{g \in \mathcal{G}^c} P(G_i = g) \sum_{g' \in \mathcal{G}^s} M_{g,g'} \cdot \mathbb{E}[h(Z_i)D_g(Z_i)] \\ &= \sum_{g \in \mathcal{G}^c} P(G_i = g) \sum_{g' \in \mathcal{G}^s} M_{g,g'} \cdot \mathbb{E}[c(g(S), Z_i)] = \sum_{g \in \mathcal{G}^c} P(G_i = g) \cdot \mathbb{E}[c(g, Z_i)] \\ &= \sum_{g \in \mathcal{G}} P(G_i = g) \cdot \mathbb{E}[c(g, Z_i)] = \mathbb{E}[c(G_i, Z_i)] = P(C_i = 1)\end{aligned}$$

where in the first step I have used that $\mathbb{E}[h(Z_i)D_g(Z_i)] = 0$ for g corresponding to always and never-takers.

E Proof of Theorem 3

E.1 Background on marginal treatment response functions

Introducing the latent indices U_{ji}

To begin, it is necessary to introduce a set of random variables $U_{1i}, U_{2i} \dots U_{ji}$, which MTW2 use to define the target causal parameters in their analysis of identification. For each $j = 1, \dots J$, the latent index U_{ji} describes individual i's selection behavior with respect to instrument Z_j when this instrument is varied in isolation. The U_{ji} can be distinguished from the selection groups G_i used to define Δ_c: instead G_i characterizes how unit i would respond to *any* counterfactual variation in the instruments, including when the values of multiple instruments are changed simultaneously. For example, with two binary instruments the ACLATE conditions on all units for whom $D_i(1, 1) > D_i(0, 0)$.

MTW2 consider target parameters that condition on the value of U_{ji} for one j at a time, while the other $J-1$ instruments held fixed at their realized values. In particular, they assume that the target parameter takes the form:

$$\beta^*(m) := \sum_{j=1}^{J}\sum_{d=0}^{1}\mathbb{E}\left[\int_0^1 \omega_j^*(d|u,Z_i)\cdot m_j(d|u,\mathbf{z}_{-j})\cdot du\right] \tag{7}$$

for some set of weights $\omega_j^*(d|u,\mathbf{z})$, where $m_j(d|u,\mathbf{z}_{-j}) := \mathbb{E}[Y_i(d)|U_{ji}=u, Z_{-ji}=\mathbf{z}_{-j}]$ is referred to as a marginal treatment response (MTR) function. Given a fixed ω^*, the parameter of interest $\beta^*(m)$ is expressed as a functional of the collection m of these MTR curves $m_j(d|\cdot,\mathbf{z}_{-j})$ across $j,d,\mathbf{z}_{-j}$. MTW2 describe how the latent indices U_{ji} can be defined by applying an equivalence result between IAM and latent index models proposed by Vytlacil (2002), if one begins by Assuming PM holds.

In particular, for each $j=1\dots J$, the variable U_{ji} can be defined in such a way that:

$$D_i(z_j, Z_{-j,i}) = \mathbb{1}\left(U_{ji} \le \mathcal{P}(z_j, Z_{-j,i})\right) \tag{8}$$

where $\mathcal{P}(\mathbf{z}) := \mathbb{E}[D_i|Z_i=\mathbf{z}]$ is the propensity score function. By (8), U_{ji} characterizes unit i's selection behavior with respect to instrument j, given that unit's realized values $Z_{-j,i}$ of the remaining instruments. If we define a function $D_{ji}(z_j) := D_i(z_j, Z_{-j,i})$ describing unit i's selection behavior as instrument Z_j is varied alone, knowledge of U_{ji} implies knowledge of $D_{ji}(\cdot)$. The latent index U_{ji} has the property that $\{U_{ji} \perp\!\!\!\perp Z_{ji}\}|Z_{-j,i}$ and can also be chosen to be continuously distributed with $U_{ji}|Z_{-j,i} \sim Unif[0,1]$. With this normalization, when $\mathcal{Z}$ is rectangular and $\mathcal{Z}_{-j}$ has the same cardinality for all j, we can think of m as a set of $2\times|\mathcal{Z}_{-j}|\times J$ functions $m_j(d|\cdot,\mathbf{z}_{-j}):[0,1]\to\mathbb{R}$.

Some care is required in comparing my target parameters to those of MTW2, since an application of the equivalence result of Vytlacil (2002) separately in each cell of $Z_{-j,i}$ does not yield a complete mapping between latent indices U_{ji} used by MTW2 and the selection groups G_i. Put another way, a unit's underlying selection group G_i is not generally pinned down from $(U_{1i},\dots,U_{Ji})$ for each j as well as Z_i, while in the other direction, Eq. (8) only requires each U_{ji} to be within a particular *range* of values (defined by the propensity score function given G_i and Z_i). I show in the next section however that parameters of the form Δ_c which are point identified can be written in terms of MTR functions in a way that is invariant over all possible choices of how U_{ji} and G_i are related within a cell of $Z_{-j,i}$, so long as the are related in such a way that Eq. (8) is satisfied. Therefore, a complete mapping between the $(U_{1i},\dots,U_{Ji},Z_i)$ and G_i is not necessary to establish my main equivalence result.[1]

[1] A complete model would be a DGP yielding the joint distribution of $(Y_i(1),Y_i(0),G_i,U_{1i}\dots U_{Ji},Z_i)$. A particularly simple such mapping would be the following. Suppose that each $\mathcal{Z}_j$ is the integers from 0 to some M_j (e.g. $M_j=1$ with binary instruments). Now let $U_{ji}=(1-W_i)\cdot\tilde{U}_{ji}+W_i\cdot\mathcal{P}(Z^*_{ji}-1,Z_{-j,i})$ where $\tilde{U}_{ji} := \mathcal{P}(Z^*_{ji},Z_{-j,i})$ with $Z^*_{ji}=\inf\{z: D_i(z,Z_{-ji})=1\}$ and $\mathcal{P}(-1,\mathbf{z}_{-j}):=0$. Take any $W_i\sim Unif[0,1]$

$\mathcal{M}^{obs}(\mathcal{S})$ and $\mathcal{M}^{lc}(\mathcal{S})$ as a system of linear equalities for the MTR curves

MTW2's notion of mutual consistency observes that because any given individual i is described by latent indices U_{ji} for *each* of the instruments $j = 1, 2, \ldots J$, that individual's potential outcome $Y_i(d)$ shows up in J different MTR curves: $m_1(d|U_{1i}, Z_{-1,i})$, $m_2(d|U_{2i}, Z_{-2,i})$, and so on to $m_J(d|U_{Ji}, Z_{-J,i})$. This means that a set of MTR functions m could be incoherent across values $j \in \{1 \ldots J\}$ and cells of $\mathbf{z}_{-j}$. To deal with this possibility in their analysis of identification, MTW2 introduce for each $s \in \mathcal{S}$ and $d \in \{0, 1\}$ the quantity $\mathbb{E}[s(D_i, Z_i)Y_i(d)]$, which is exactly analogous to β_s but replaces unit i's realized value of the outcome Y_i with a potential outcome $Y_i(d)$. Mutual consistency requires the β_s to have equal values when represented in terms of any particular instrument j's MTR functions (Eq. 22 of MTW2):

$$\mathbb{E}[s(D_i, Z_i)Y_i(d)] = \mathbb{E}\left[\int_0^1 \{s(1, Z_i)\mathbb{1}(u \leq \mathcal{P}(Z_i)) + s(0, Z_i)\mathbb{1}(u > \mathcal{P}(Z_i))\} \cdot m_j(d|u, Z_{-j,i}) \cdot du\right], \tag{9}$$

Eq. (9) can be obtained by the law of iterated expectations over Z_{-j} and U_j. Mutual consistency says that the MTR functions m are such that the RHS of (9) is the same for all $j \in \{1 \ldots J\}$, i.e.

$$\mathcal{M}^{lc(\mathcal{S})} = \{m : \text{ the RHS of (9) doesn't depend on } j, \forall s \in \mathcal{S}\}$$

Mogstad et al. (2018) show that the corresponding IV-like estimand β_s can be expressed similarly in terms of the MTR functions as:

$$\beta_s = \mathbb{E}\left[\int_0^1 \{s(1, Z_i)\mathbb{1}(u \leq \mathcal{P}(Z_i))m_j(1|u, Z_{-j,i}) + s(0, Z_i)\mathbb{1}(u > \mathcal{P}(Z_i))m_j(0|u, Z_{-j,i})\} \cdot du\right] \tag{10}$$

Given this, we can write

$$\mathcal{M}^{obs(\mathcal{S})} = \{m \text{ for which (10) holds }, \forall s \in \mathcal{S}\}$$

Comparing (9) and (10), Eq. (9) considers the MTR function for $Y_i(d)$ for all u and Z_i, while (10) considers the treatment status implied by a given value of j, $Z_{-j,i}$ and U_{ji}.

Deriving the weights $\omega_j(d|u, \mathbf{z})$ for Δ_c satisfying Property M

Now turn to the setting of Theorem 3 in which Assumptions 1-3 hold with J binary instruments. Consider a Δ_c of the form

$$\Delta_c = \mathbb{E}\left[Y_i(1) - Y_i(0) \,\middle|\, \bigcup_{k=1}^{K} \{i : D_i(\mathbf{u}_k(Z_i)) > D_i(\mathbf{l}_k(Z_i))\}\right]$$

where $W_i \perp\!\!\!\perp (G_i, Z_i, Y_i(1), Y_i(0))$. With this definition we have the standard assumptions for a latent-index IV model with random assignment conditional on $Z_{-j,i}$ for each $j = 1 \ldots J$: $\{(U_{ji}, Y_i(1), Y_i(0)) \perp\!\!\!\perp Z_{ji}\} \,|Z_{-j,i}$, and decision-model (8): $D_{ji}(z_j) = \mathbb{1}\left(U_{ji} \leq \mathcal{P}(z_j, Z_{-j,i})\right)$ with $U_{ji}|Z_{-j,i} \sim Unif[0, 1]$.

for some sequence of functions $\mathbf{u}_k$ and $\mathbf{l}_k$ such that $\mathbf{u}_k(\mathbf{z}) \geq \mathbf{l}_k(\mathbf{z}) \geq \mathbf{u}_{k+1}(\mathbf{z})$ component-wise, for all k and $\mathbf{z} \in \mathcal{Z}$. Given a parameter of interest, the functions $\mathbf{u}_k$ and $\mathbf{l}_k$ are known. Theorems 1 and 2 combined with Proposition 3 of this paper show that parameters of the form Δ_c are point identified with binary instruments with full support satisfying VM if and only if they can be written in the above form.

Since Assumption 3 implies that $\mathcal{Z}$ is non-disjoint (as defined in Proposition 1), then any two vectors that are ordered component-wise can be linked by a "chain" of vectors in which each differs from the next in only one component. For example, $\mathbf{l}_k(\mathbf{z}) = (1,1)'$ and $u_k(\mathbf{z}) = (0,0)$ can be linked by the chain $(0,0),(0,1),(1,1)$ (or alternatively by $(0,0),(1,0),(1,1)$). The event $D_i(1,1) > D_i(0,0)$ is equivalent to the event that either $D_i(1,1) > D_i(0,1)$ or that $D_i(0,1) > D_i(0,0)$. Thus, without loss of generality we can suppose that for each $\mathbf{z}$, $\mathbf{u}_k(\mathbf{z})$ and $\mathbf{l}_k(\mathbf{z})$ differ only in that one particular instrument takes a "1" value in $\mathbf{u}_k(\mathbf{z})$ and a "zero" value in $\mathbf{l}_k(\mathbf{z})$. Let $j_k(\mathbf{z})$ be the instrument that shifts from zero to one between $\mathbf{l}_k(\mathbf{z})$ and $\mathbf{u}_k(\mathbf{z})$, with K the maximum necessary number of such switches for this representation to hold over all $\mathbf{z}$.

Since for a given i, we can only have $D_i(\mathbf{u}_k(\mathbf{z})) > D_i(\mathbf{l}_k(\mathbf{z}))$ for one value of k, we can write, defining the shorthand $P(\mathbf{z}) = P(Z_i = \mathbf{z})$ and using $P(\mathbf{z}|C_i = 1) = P(Z_i = \mathbf{z})P(c(G_i, \mathbf{z}) = 1)/P(C_i = 1)$ by Assumption 1:

$$\Delta_c = \sum_{\mathbf{z}} P(\mathbf{z}) \cdot \sum_{k=1}^{K} \frac{P(D_i(\mathbf{u}_k(\mathbf{z})) > D_i(\mathbf{l}_k(\mathbf{z})))}{P(C_i = 1)} \cdot \mathbb{E}\left[Y_i(1) - Y_i(0)\,|D_i(\mathbf{u}_k(\mathbf{z})) > D_i(\mathbf{l}_k(\mathbf{z}))\right] \tag{11}$$

where $P(C_i = 1) = \sum_{\mathbf{z}} P(\mathbf{z}) \cdot \sum_{k=1}^{K} P(D_i(\mathbf{u}_k(\mathbf{z})) > D_i(\mathbf{l}_k(\mathbf{z}))) = \sum_{k=1}^{K} \mathbb{E}\left[\mathcal{P}(\mathbf{u}_k(Z_i)) - \mathcal{P}(\mathbf{l}_k(Z_i))\right]$. Given that only one instrument is changed within in each $(\mathbf{z}, k)$ term above, we can express Δ_c in terms of the MTR functions m. By (8), note that an individual with $Z_{-j,i} = \mathbf{z}_{-j}$ has a value of U_{ji} between $\mathcal{P}(0, \mathbf{z}_{-j})$ and $\mathcal{P}(1, \mathbf{z}_{-j})$ if and only if $D_i(1, \mathbf{z}_{-j}) > D_i(0, \mathbf{z}_{-j})$, and thus

$$\begin{aligned}\mathbb{E}[Y_i(1) - Y_i(0)|D_i(1, \mathbf{z}_{-j}) > D_i(0, \mathbf{z}_{-j})] &= \mathbb{E}[Y_i(1) - Y_i(0)|\mathcal{P}(0, \mathbf{z}_{-j}) \leq U_{ji} < \mathcal{P}(1, \mathbf{z}_{-j}), Z_{-j,i} = \mathbf{z}_{-j}] \\ &= \frac{1}{\mathcal{P}(1, \mathbf{z}_{-j}) - \mathcal{P}(0, \mathbf{z}_{-j})} \sum_{d=0}^{1} (-1)^{d+1} \int_{\mathcal{P}(0,\mathbf{z}_{-j})}^{\mathcal{P}(1,\mathbf{z}_{-j})} m_j(d|u, \mathbf{z}_{-j}) \cdot du\end{aligned}$$

using that $U_{ji}|Z_{-j,i} \sim Unif[0,1]$ and $Z_{-j,i} \perp\!\!\!\perp (G_i, Y_i(1), Y_i(0))$.

Thus, organizing the terms of (11) by which instrument is varied in each term, we have:

$$\Delta_c = \sum_{\mathbf{z}} \frac{P(\mathbf{z})}{P(C_i = 1)} \cdot \sum_{j=1}^{J} \sum_{k=1}^{K} \mathbb{1}(j = j_k(\mathbf{z})) \cdot \sum_{d=0}^{1} (-1)^{d+1} \int_{\mathcal{P}(\mathbf{l}_k(\mathbf{z}))}^{\mathcal{P}(\mathbf{u}_k(\mathbf{z}))} m_j(d|u, \mathbf{l}_{k,-j}(\mathbf{z})) \cdot du \tag{12}$$

where $\mathbf{l}_{k,-j}(\mathbf{z})$ are the $J-1$ components of $\mathbf{l}_k(\mathbf{z})$ corresponding to the other $J-1$ instruments aside from $j_k(\mathbf{z})$, and I've used that $\mathbf{u}_k(\mathbf{z}) = (1, \mathbf{l}_{k,-j}(\mathbf{z}))$ and $\mathbf{l}_k(\mathbf{z}) = (0, \mathbf{l}_{k,-j}(\mathbf{z}))$. Then, comparing with Eq. (7), $\Delta_c = \beta^*(m)$ where:[2]

$$\omega_j(d|u,\mathbf{z}) = (-1)^{d+1}\sum_{k=1}^{K}\cdot\frac{\mathbb{1}(j=j_k(\mathbf{z}))\cdot\mathbb{1}(\mathcal{P}(\mathbf{u}_k(\mathbf{z}))\geq u\geq\mathcal{P}(\mathbf{l}_k(\mathbf{z})))}{\sum_{k'=1}^{K}\mathbb{E}\left[\mathcal{P}(\mathbf{u}_{k'}(Z_i))-\mathcal{P}(\mathbf{l}_{k'}(Z_i))\right]} \tag{13}$$

E.2 Proving Theorem 3

Preliminary: "concordance" between MTR curves m and $\mathbf{x}$

The results in this paper characterize identification of parameters of the form Δ_c by expressing them as linear functions of the $2|\mathcal{G}|$ component vector $\mathbf{x} = (\mathbf{x}(0)', \mathbf{x}(1)')'$, with components $x_{dg} = x(d)_g := \mathbb{E}[Y_i(d)|G_i = g]$, for $d \in \{0,1\}$ and $g \in \mathcal{G}$. To characterize the set $\mathcal{M}^{MTW}(\bar{\mathcal{S}})$ defined in terms of MTR functions m delivered by MTW2's approach, I will find it useful to rewrite the set $\mathcal{M}^{MTW}(\bar{\mathcal{S}})$ in terms of $\mathbf{x}$ instead of m. To do this, I first formalize a notion in which a particular m and a particular value of $\mathbf{x}$ can describe the same DGP, given that Eq. (8) must hold. This is necessary (absent a complete model in the sense of Footnote 1) because the response groups G_i are not pinned down by latent indices U_{ji} and vice-versa. I make this precise through a notion I call "concordance" between a given m and a given $\mathbf{x}$.

Note that for any $\mathbf{z} \in \mathcal{Z}, g \in \mathcal{G}$ and $d, d' \in \{0,1\}$, Eq. (8) implies that if we sum over $x_{d'g}$ for groups sharing value d of $D_g(\mathbf{z})$, we can write this in terms of MTR functions m as:

$$\begin{aligned}\sum_{g\in\mathcal{G}}\mathbb{1}(D_g(\mathbf{z})=d)\cdot x_{d'g} &= P(G_i\in\{g:D_g(\mathbf{z})=d\})\cdot\mathbb{E}[Y_i(d')|G_i\in\{g:D_g(\mathbf{z})=d\}]\\ &= \begin{cases}\int_{u\leq\mathcal{P}(1,\mathbf{z}_{-j})} m_j(d'|u,\mathbf{z}_{-j})\cdot du & \text{if } d=1\\ \int_{u\geq\mathcal{P}(0,\mathbf{z}_{-j})} m_j(d'|u,\mathbf{z}_{-j})\cdot du & \text{if } d=0\end{cases} = \int_{u:(-1)^d\cdot u\geq(-1)^d\cdot\mathcal{P}(d,\mathbf{z}_{-j})} m_j(d'|u,\mathbf{z}_{-j})\cdot du\end{aligned} \tag{14}$$

When $d = d'$, we have the simpler expression:

$$\sum_{g\in\mathcal{G}}\mathbb{1}(D_g(\mathbf{z})=d)\cdot x_{dg} = \int_{u:(-1)^d\cdot u\leq(-1)^d\cdot\mathcal{P}(d,\mathbf{z}_{-j})} m_j(d|u,\mathbf{z}_{-j})\cdot du \tag{15}$$

For any m and $\mathbf{x} \in \mathbb{R}^{2|\mathcal{G}|}$, call the pair $(m, \mathbf{x})$ *concordant* if Eq. (15) holds for all $j = 1 \dots J$, $d \in \{0,1\}$ and $\mathbf{z} \in \mathcal{Z}$. Let us denote the set of concordant pairs $(m, \mathbf{x})$ as $\Phi(\mathcal{G}, \mathcal{P})$. This set depends upon the propensity score function $\mathcal{P}$ and the support $\mathcal{G}$ of response types.[3] Here $\mathcal{G}$

[2] As a concrete example, consider the ACLATE with two binary instruments. Letting $\mathbf{l}_1(\mathbf{z}) = (0,0), \mathbf{u}_1(\mathbf{z}) = (1,0), \mathbf{l}_2(\mathbf{z}) = (1,0), \mathbf{u}_2(\mathbf{z}) = (1,1)$, we have that $j_1(\mathbf{z}) = 1$ and $j_1(\mathbf{z}) = 2$. Then:

$$\omega_1(d|u,\mathbf{z}) = (-1)^{d+1}\cdot\frac{\mathbb{1}(\mathcal{P}(1,0)\geq u\geq\mathcal{P}(0,0)}{\mathcal{P}(1,1)-\mathcal{P}(0,0)}, \qquad \omega_2(d|u,\mathbf{z}) = (-1)^{d+1}\cdot\frac{\mathbb{1}(\mathcal{P}(1,1)\geq u\geq\mathcal{P}(1,0)}{\mathcal{P}(1,1)-\mathcal{P}(0,0)}$$

[3] Recall that there is only one propensity score function $\mathcal{P}(\cdot)$ compatible with a given $\mathcal{P}_{DZ}$ and Assumption 1, where $\mathcal{P}_{DZ}$ denotes the observable joint distribution of D_i and Z_i. In general the set $\Phi(\mathcal{G}, \mathcal{P})$ would also be a function of the $\mathcal{Z}_j$ for $j = 1 \dots J$, but I leave this implicit given that I will maintain Assumption 3.

denotes these types that occur in the actual DGP, rather than restrictions on the response types that might be assumed by the researcher (e.g. VM vs. PM).

The notion of concordant pairs proves useful in relating $B^{MTW}(\bar{\mathcal{S}})$ with the set $\{\theta_c'\mathbf{x} : A^{VM}\mathbf{x} = \mathbf{b}\}$, to establish Theorem 3. Let $\mathcal{G}^{VM}$ denote the set of Ded_J response groups that exist under VM with J binary instruments. For any $\mathbf{b} \in \mathbb{R}^{2 \cdot 2^J}$, let

$$\mathcal{M}^{lin(\mathcal{P},\mathbf{b})} := \{m : (m, \mathbf{x}) \in \Phi(\mathcal{G}^{VM}, \mathcal{P}) \text{ for an } \mathbf{x} \in \mathbb{R}^{2 \cdot Ded_J} \text{ satisfying } A^{VM}\mathbf{x} = \mathbf{b}\}$$

be the set of MTR collections m that are concordant with some $\mathbf{x}$ that solves $A^{VM}\mathbf{x} = \mathbf{b}$. The set $\mathcal{M}^{lin(\mathcal{P},\mathbf{b})}$ depends on the propensity score function $\mathcal{P}$ and $\mathbf{b}$.

With this background in place, I turn to proving Theorem 3 maintaining Assumptions 1-3.

Step 1: $\mathcal{M}^{obs(\bar{\mathcal{S}})} \subseteq \mathcal{M}^{lin(\mathcal{P},\mathbf{b})}$

This section shows that if $m \in \mathcal{M}^{obs(\bar{\mathcal{S}})}$, then $m \in \mathcal{M}^{lin(\mathcal{P},\mathbf{b})}$. Consider any $m \notin \mathcal{M}^{lin(\mathcal{P},\mathbf{b})}$. Then for all $\mathbf{x} \in \mathbb{R}^{2 \cdot Ded_J}$ such that $(m, \mathbf{x}) \in \Phi(\mathcal{G}, \mathcal{P})$, it must be the case that $A^{VM}\mathbf{x} \neq \mathbf{b}$. Let $\mathbf{x}^*$ be the value of $\mathbf{x}$ according to the true DGP. Since $A^{VM}\mathbf{x}^* = \mathbf{b}$ given Assumptions 1-2 (i.e. that vector monotonicity in fact holds with valid instruments), this implies that m and $\mathbf{x}^*$ are not concordant, i.e. $\sum_{g \in \mathcal{G}} \mathbb{1}(D_g(\mathbf{z}) = d) \cdot x_{dg}^* \neq \int_{u:(-1)^d \cdot u \leq (-1)^d \cdot \mathcal{P}(d, \mathbf{z}_{-j})} m_j(d|u, \mathbf{z}_{-j}) \cdot du$ for some $d \in \{0, 1\}, \mathbf{z} \in \{0, 1\}^J$ and j. This in turn implies that:

$$b_{d\mathbf{z}} \neq \int_{u:(-1)^d \cdot u \leq (-1)^d \cdot \mathcal{P}(d, \mathbf{z}_{-j})} m_j(d|u, \mathbf{z}_{-j}) \cdot du,$$

since we can write $A^{VM}\mathbf{x}^* = \mathbf{b}$ as $\sum_{g \in \mathcal{G}} \mathbb{1}(D_g(\mathbf{z}) = d) \cdot x_{dg}^* = b_{d\mathbf{z}}$ (for all $d \in \{0, 1\}, \mathbf{z} \in \{0, 1\}^J$). Now since $\beta_{s_{d,\mathbf{z}}} = P(Z_i = \mathbf{z}) \cdot b_{d\mathbf{z}}$ and $P(Z_i = \mathbf{z}) \neq 0$ for any $\mathbf{z} \in \{0, 1\}^J$ given Assumption 3, this requires that:

$$\beta_{s_{d,\mathbf{z}}} \neq P(Z_i = \mathbf{z}) \cdot \int_{u:(-1)^d \cdot u \leq (-1)^d \cdot \mathcal{P}(d, \mathbf{z}_{-j})} m_j(d|u, \mathbf{z}_{-j}) \cdot du$$

But the RHS above is the expression for $\beta_{s_{d,\mathbf{z}}}$ given by (10). Thus, m must violate (10) for some $s_{d,\mathbf{z}} \in \bar{\mathcal{S}}$ and j, and we can therefore conclude that $m \notin \mathcal{M}^{obs(\bar{\mathcal{S}})}$.

Step 2: $(m, \mathbf{x}) \in \Phi(\mathcal{G}, \mathcal{P})$ **implies** $\beta^*(m) = \theta_c'\mathbf{x}$ **under VM**

The next piece is to note that for any Δ_c satisfying Property M, if $(m, \mathbf{x}) \in \Phi(\mathcal{G}, \mathcal{P})$, then $\beta^*(m)$ must be equal to $\theta_c'\mathbf{x}$ if VM holds. To show this, observe that when VM holds $\mathcal{P}(1, \mathbf{z}_{-j}) \geq \mathcal{P}(0, \mathbf{z}_{-j})$ for any j and $\mathbf{z}_{-j} \in \mathcal{Z}_{-j}$ (given our labeling of the values of Z_j), and

$$\mathbb{1}(D_g(1, \mathbf{z}_{-j}) > D_g(0, \mathbf{z}_{-j})) = \mathbb{1}(D_g(\mathbf{z}) = 1) - \mathbb{1}(D_g(\mathbf{z}) = 0)$$

The concordance condition Eq. (15) then implies that

$$\sum_{g\in\mathcal{G}} \mathbb{1}(D_g(1,\mathbf{z}_{-j}) > D_g(0,\mathbf{z}_{-j}))\cdot x_{dg} = \int_{\mathcal{P}(0,\mathbf{z}_{-j})}^{\mathcal{P}(1,\mathbf{z}_{-j})} m_j(d|u,\mathbf{z}_{-j})\cdot du \tag{16}$$

Now, combining (7) with (13) (which holds for Δ_c satisfying Property M), we can use (16) to write:

$$\begin{aligned}
\beta^*(m) &= \sum_{\mathbf{z}} P(Z_i=\mathbf{z})\cdot\sum_{j=1}^{J}\sum_{k=1}^{K}\mathbb{1}(j=j_k(\mathbf{z}))\cdot\frac{\sum_{d=0}^{1}(-1)^{d+1}\int_{\mathcal{P}(\mathbf{l}_k(\mathbf{z}))}^{\mathcal{P}(\mathbf{u}_k(\mathbf{z}))} m_j(d|u,\mathbf{l}_{k,-j}(\mathbf{z}))\cdot du}{\sum_{k'=1}^{K}\mathbb{E}\left[\mathcal{P}(\mathbf{u}_{k'}(Z_i))-\mathcal{P}(\mathbf{l}_{k'}(Z_i))\right]}\\
&= \sum_{\mathbf{z}} P(Z_i=\mathbf{z})\cdot\sum_{k=1}^{K}\mathbb{1}(j=j_k(\mathbf{z}))\cdot\frac{\sum_{d=0}^{1}(-1)^{d+1}\sum_{g\in\mathcal{G}}\mathbb{1}(D_g(\mathbf{u}_k(\mathbf{z}))>D_g(\mathbf{l}_k(\mathbf{z})))\cdot x_{dg}}{\sum_{k'=1}^{K}\mathbb{E}\left[\mathcal{P}(\mathbf{u}_{k'}(Z_i))-\mathcal{P}(\mathbf{l}_{k'}(Z_i))\right]}\\
&= \sum_{d=0}^{1}(-1)^{d+1}\cdot\frac{\sum_{g\in\mathcal{G}}\sum_{\mathbf{z}}P(Z_i=\mathbf{z})\sum_{k=1}^{K}\mathbb{1}(D_g(\mathbf{u}_k(\mathbf{z}))>D_g(\mathbf{l}_k(\mathbf{z}))\cdot x_{dg}}{\sum_{k'=1}^{K}\mathbb{E}\left[\mathcal{P}(\mathbf{u}_{k'}(Z_i))-\mathcal{P}(\mathbf{l}_{k'}(Z_i))\right]}\\
&= \sum_{d=0}^{1}(-1)^{d+1}\cdot\frac{\sum_{g\in\mathcal{G}}\sum_{\mathbf{z}}P(Z_i=\mathbf{z})\cdot\mathbb{1}\left(\sum_{k=1}^{K}D_g(\mathbf{u}_k(\mathbf{z}))>D_g(\mathbf{l}_k(\mathbf{z})\right)\cdot x_{dg}}{\mathbb{E}\left[\sum_{k'=1}^{K}\mathcal{P}(\mathbf{u}_{k'}(Z_i))-\mathcal{P}(\mathbf{l}_{k'}(Z_i))\right]}\\
&= \sum_{d=0}^{1}(-1)^{d+1}\cdot\frac{\sum_{g\in\mathcal{G}}\mathbb{E}\left[\sum_{k=1}^{K}D_g(\mathbf{u}_k(\mathbf{Z_i}))>D_g(\mathbf{l}_k(\mathbf{Z_i})\right]\cdot x_{dg}}{\mathbb{E}[c(G_i,Z_i)]}\\
&= \sum_{d=0}^{1}(-1)^{d+1}\cdot\sum_{g\in\mathcal{G}}\frac{\mathbb{E}[c(g,Z_i)]}{P(C_i=1)}\cdot x_{dg} = \theta_c'\mathbf{x}
\end{aligned}$$

Step 3: an outer set for $B^{MTW}(\bar{\mathcal{S}})$ in terms of the vector x

Recall that

$$B^{MTW}(\mathcal{S}) := \{\beta^*(m) : m\in(\mathcal{M}\cap\mathcal{M}^{obs(\mathcal{S})}\cap\mathcal{M}^{lc(\mathcal{S})})\}$$

Consider the following outer set for $B^{MTW}(\mathcal{S})$

$$B^{outer}(\mathcal{S}) := \{\beta(m) : m\in\mathcal{M}^{obs(\mathcal{S})}\}$$

where no additional assumptions about the MTR curves are leveraged via $\mathcal{M}$, nor is MTW2's notion of mutual consistency imposed. Clearly $B^{MTW}(\mathcal{S})\subseteq B^{outer}(\mathcal{S})$ for any $\mathcal{S}$.

Together, Steps 1 and 2 above show that for any $m\in\mathcal{M}^{obs(\bar{\mathcal{S}})}$, $\beta^*(m)=\theta_c'\mathbf{x}$ where $\mathbf{x}\in\mathbb{R}^{2\cdot Ded_J}$ is a solution to $A^{VM}\mathbf{x}=\mathbf{b}$. We then have that

$$B^{outer}(\bar{\mathcal{S}})\subseteq\{\theta_{\mathbf{c}}{}'\mathbf{x} : A^{VM}x=\mathbf{b}\} \tag{17}$$

which establishes Theorem 3, since $B^{MTW}(\bar{\mathcal{S}})\subseteq B^{outer}(\bar{\mathcal{S}})$.

Remark on mutual consistency

Note that we did not need to make use of MTW2's notion of mutual consistency to establish Theorem 3. The mutual consistency property (9) applied to any $s_{d,\mathbf{z}} \in \bar{\mathcal{S}}$ says that

$$\mathbb{E}[s_{d,\mathbf{z}}(D_i, Z_i)Y_i(d')] = P(Z_i = \mathbf{z}) \cdot \int_{u:(-1)^d \cdot u \geq (-1)^d \cdot \mathcal{P}(d,\mathbf{z}_{-j})} m_j(d'|u, \mathbf{z}_{-j}) \cdot du$$

is the same for all j, for any $\mathbf{z} \in \mathcal{Z}$, $d, d' \in \{0, 1\}$. Equivalently, for any $\mathbf{z} \in \mathcal{Z}$, $d, d' \in \{0, 1\}$:

$$\int_{u:(-1)^d \cdot u \geq (-1)^d \cdot \mathcal{P}(d,\mathbf{z}_{-j})} m_j(d'|u, \mathbf{z}_{-j}) \cdot du = \int_{u:(-1)^d \cdot u \geq (-1)^d \cdot \mathcal{P}(d,\mathbf{z}_{-j})} m'_j(d'|u, \mathbf{z}_{-j'}) \cdot du \qquad (18)$$

for all $j' \neq j$. Recall Eq. (14), which derived an implication of Eq. (8) that relates m and $\mathbf{x}$ but is more general than the relationship used to define concordance (which took the special case of $d = d'$). Eq. (14) implies (18) already, since the LHS of (14) does not depend on j in any way. Thus mutual consistency becomes unnecessary to impose once $\beta^*(m)$ is expressed in terms of the vector $\mathbf{x}$ rather than in terms of single-instrument MTR curves m.

Additional References

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